

# THE DETERMINATION OF THE TRACK MODULUS $k$ FOR THE STANDARD TRACK ANALYSIS

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## ***Abstract***

After the adoption of the equation  $EIw^{iv} + kw = q$  for cross-tie track analyses, a number of methods for the determination of the track modulus  $k$  were proposed and used during the past several decades. However, some of these methods are difficult to utilize or are of questionable validity. Therefore, at first, a simple yet accurate method is presented and its use is illustrated on practical examples. Then, other published methods are critically reviewed and their shortcomings are pointed out.

## ***1. Introduction***

The analysis for cross-tie tracks, recommended in the AREA Manual (1991, Section 22), assumes that the rail responds like an elastic beam that is attached to a continuous base of closely spaced elastic springs, as shown in Fig. 1. For a description of the historical evolution of track stress analyses refer to Kerr (1976).

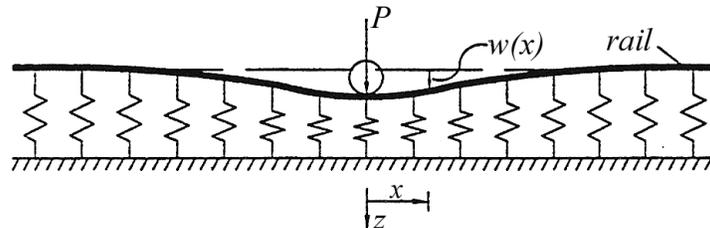


Fig. 1 Rail-in-Track Subjected to a Wheel Load

The corresponding governing equation is

$$EI \frac{d^4 w}{dx^4} + kw(x) = q(x) \quad (1)$$

in which  $w(x)$  is the vertical deflection of the rail axis at  $x$ ,  $EI$  is the vertical flexural stiffness of one rail,  $k$  is the track modulus (for one rail), and  $q(x)$  represents the vertical wheel loads.

For one wheel load  $P$ , the deflection profile is obtained as

$$w(x) = \frac{P\beta}{2k} e^{-\beta x} (\cos \beta x + \sin \beta x) \quad (2)$$

where

$$\beta = \sqrt[4]{\frac{k}{4EI}}.$$

For more than one wheel load, the rail deflections may be obtained by superposing the effects of the various wheel loads.

For a track analysis, the parameters that enter the expression in (2) are needed.  $E$  is Young's modulus of rail steel and is known,  $I$  is the bending moment of inertia of the rail under consideration and is listed in the AREA Manual (1991, Chapter 4), and  $P$  is a known wheel load. *The only unknown is the track modulus  $k$ .*

The purpose of the present paper is to discuss the various proposed methods for the determination of  $k$ .

## 2. An Early Method for the Determination of $k$

In this approach, promoted by the research of Timoshenko and Langer (1932), the used loading device consisted of *one axle*. The rail deflection at the wheel,  $w_m$ , caused by wheel load  $P$ , is recorded and then collocated (i.e. equated) with the corresponding analytical expression obtained from eq. (2) at  $x=0$ ; namely, by setting  $w_m = w(0)$ . The resulting equation is

$$w_m = \frac{P\beta}{2k} = \frac{P \sqrt[4]{\frac{k}{4EI}}}{2k}. \quad (3)$$

Solving it for  $k$ , the only unknown, yields

$$k = \frac{1}{4} \sqrt[3]{\frac{P^4}{EIw_m^4}}; \quad (4)$$

an explicit expression for the track modulus.

As an *example* consider a wood-tie track with 136 RE rails loaded by one axle with a wheel load  $P = 30,000$  lb = 15 tons. The recorded deflection caused by  $P$  is  $w_m = 0.12$  inches. According to eq. (4) the corresponding track modulus (for one rail) is

$$k = \frac{1}{4} \sqrt[3]{\frac{(30,000)^4}{30 \times 10^6 \times 95.00 \times (0.12)^4}} = 2,775 \text{ [lb/in}^2\text{]} .$$

The above method is very simple, since it requires only one deflection measurement and a simple calculation. Another advantage is that because of the bending stiffness of the rail, the ballast-subgrade conditions are averaged out over the affected track section. Because of its simplicity, eq. (4) is being recommended for the calculation of the track modulus  $k$ , even in the recently published texts on railroad engineering. For examples refer to Hay (1982, page. 262) and to Eisenmann in Fastenrath (1981, page 36).

The major shortcoming of using eq. (4) for the determination of the track modulus  $k$  is that it requires a special test set-up with one-axle wheel loads. One such set-up was used by the Talbot Committee (1918) for determining rail deflection profiles. It consisted of a flat car loaded with rails weighing 25 to 50 tons, and equipped with load indicating screw jacks, as shown in Fig. 2.

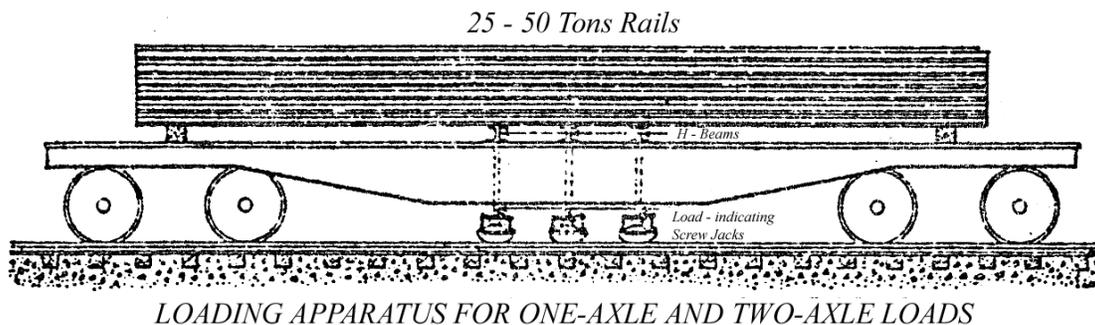


Fig. 2 Talbot Committee Loading Device (1918)

The outer jacks were used to simulate the wheel loads of a two-axle truck, whereas the middle one simulated a truck with one axle load. Cars of the same type have been used for the determination of  $k$ -values in western Europe [Driessen (1937), Birman (1957), Nagel (1961)] and in the former USSR [Kuptsov (1975)] to simulate a one-axle load. More recently, a static one-axle loading device was used by Zarembski and Choros (1980) in the AAR laboratory in Chicago. But such special one-axle loading devices are, generally, not available to railway engineers; or for that matter, not even to the majority of railway researchers.

It was therefore essential to establish a simple procedure that retains the simplicity of the above method, but is able to utilize any available car or locomotive on *two or three-axle trucks* as a loading device. This was done by Kerr (1983, 1987, 2000).

### 3. The Kerr Method for Determination of $k$ Using Any Car or Locomotive

To demonstrate this method, consider a car on *two-axle trucks*, as shown in the insert of Fig. 3. The analytical expression for the rail deflection at the left wheel of truck ① is obtained by superposition, using eq. (2). Since the wheel loads of each truck are equal but the load exerted on each truck may be different, we set

$$P_1 = P_2 = P \quad \text{and} \quad P_3 = P_4 = nP$$

where  $n$  is known. The number  $n$  is obtained by weighing; namely by placing the first truck ① and then truck ② on a track scale.

The analytical expression for the *vertical rail deflection* at the left wheel of truck ①, caused by all four wheels of the two trucks, is obtained by superposing the corresponding  $w(x)$  expressions given in eq. (2). It is, since  $l_1 = 0$ ,

$$w(0) = \frac{P\beta}{2k} + \frac{P\beta}{2k} e^{-\beta l_2} (\cos \beta l_2 + \sin \beta l_2) + \frac{nP\beta}{2k} e^{-\beta l_3} (\cos \beta l_3 + \sin \beta l_3) + \frac{nP\beta}{2k} e^{-\beta l_4} (\cos \beta l_4 + \sin \beta l_4) \quad (5)$$

where as before  $\beta = \sqrt[4]{k/(4EI)}$ .

The rail support modulus  $k$  is obtained by collocating (equating) this deflection with the deflection measured at the left wheel,  $w_m$ ; namely by setting  $w(0) = w_m$ . This yields

$$\frac{w_m}{P} = \frac{\beta}{2k} \left[ 1 + e^{-\beta l_2} (\cos \beta l_2 + \sin \beta l_2) + ne^{-\beta l_3} (\cos \beta l_3 + \sin \beta l_3) + ne^{-\beta l_4} (\cos \beta l_4 + \sin \beta l_4) \right]. \quad (6)$$

In above equation all quantities, except  $k$ , are known for a given field test. This equation is equivalent to eq. (4) for one wheel load. Whereas eq. (3) was solved explicitly for  $k$ , this is not possible for eq. (6).

To avoid involved solutions of the above transcendental equation for  $k$ , the right hand side of eq. (6) was evaluated numerically for given sets of  $(E, I, w_m, P)$  - values by substituting different values of  $k$  from the range of 500 to 9,000 lb/in<sup>2</sup>. It was assumed that the wheel distances are those of a freight car with  $l_1 = 0$ ,  $l_2 = 5'-10" = 70"$ ,  $l_3 = 46'-3"$ , and  $l_4 = 52'-1"$  (distance between truck centers 46'-3"). The results of this numerical evaluation are presented graphically in Fig. 3.

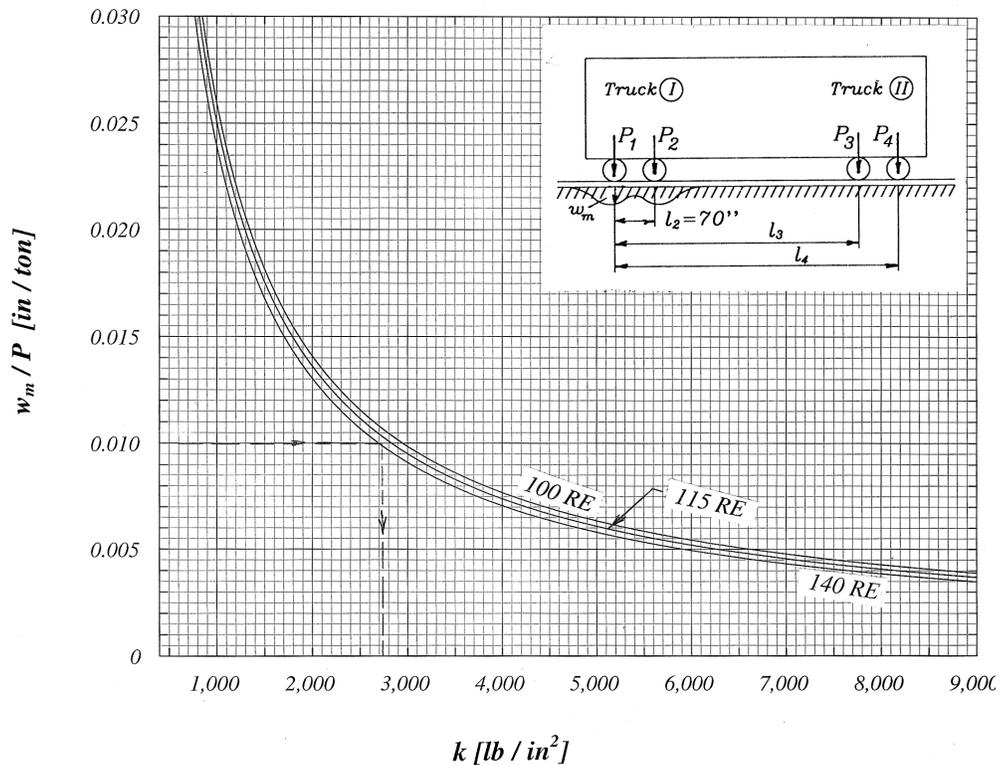


Fig. 3 Master Chart for the Determination of  $k$  Using a Vehicle on 2-Axle Trucks

To check the effect of truck ② on the results, the evaluations were conducted for  $n = 1.0$ ,  $0.5$ , and  $0$ . It was found that for the wheel distances  $l_3$  and  $l_4$  used, truck ② had no noticeable effect on the  $w_m/P$  values, even for  $k$  as low as  $1,000 \text{ lb/in}^2$ . Thus, whether the wheel loads of truck ② are equal to those of truck ① (i.e.  $n = 1$ ) or they are only about one half of those of truck ① (i.e.  $n = 0.5$ ), the curves presented in Fig. 3 are still valid. This is a useful finding, since the vertical forces a loaded car, or a locomotive, exert on their trucks generally differ.

The graphs presented in Fig. 3 are for rails 100 RE, 115 RE, and 140 RE. Those for other rail sizes were not included due to space limitation between the shown curves. However, because of the proximity of the presented curves, values for the missing rail sizes may be easily obtained by interpolation. The same argument applies to worn rails.

It is proposed to use the graphs in Fig. 3 for the determination of the  $k$  modulus, as follows: First measure the deflection  $w_m$  caused by a car on two-axle trucks with wheel loads  $P_m$ , as shown in the insert of Fig. 3. Then form  $w_m/P_m$ . The graph for the corresponding rail yields directly the  $k$ -value.

As an *example*, a loaded freight car on *two-axle trucks* is chosen as a loading device for the field test to be performed, at a track location of interest. As a first step, the wheel loads of one of the trucks, say truck ①, are determined by placing the truck on a car scale for weighing. It was found to be  $118,400 \text{ lb}$ . Assuming that each of the four wheels in truck ① carries approximately the same load, the wheel load  $P_m$  is calculated as

$$P_m = 118,400/4 = 29,600 \text{ lb} = 14.8 \text{ tons} \quad (7)$$

Next, a fine scale equipped with a magnet is attached vertically to the rail web, at the track location of interest. Then the test car is moved to this location. When the front wheel of truck ① reaches the point above the scale, the vertical rail deflection is recorded using a level placed about 30 feet from the rail; say  $w_m = 0.15$  inches. The ratio  $w_m/P_m$  is then formed; namely

$$\frac{w_m}{P_m} = \frac{0.15}{14.8} = 0.0101 \text{ in/ton} \quad (8)$$

The test was conducted on a track with 132 RE rails that showed minor wear. With  $w_m/P_m = 0.0101$  in/ton, the graphs in Fig. 3 yield directly

$$k \cong 2,730 \text{ lb/in}^2 . \quad (9)$$

This completes the determination of  $k$  at this location. Note, that by using the graphs in Fig. 3, the track modulus  $k$  is obtained for a given  $w_m/P_m$ -value without any additional calculations.

To determine the  $k$ -value at another location, move the fine scale and then the test car to the new location, measure  $w_m$ , calculate  $w_m/P_m$ , and get the corresponding  $k$  value from Fig. 3.

The procedure for determining the track modulus using a locomotive on two-axle trucks is the same as the one discussed above, except that eq. (6) has to be evaluated for different values of the axle spaces  $l_2, l_3, l_4$ , if the two wheel loads of truck  $\textcircled{1}$  are the same.

The graphs in Fig. 3 exhibit an interesting feature. In discussing the governing equation (1), it was stated that the  $k$ -value (the stiffness of the elastic spring layer) represents the response of the base *under* the rail; thus, of the cross-ties, fasteners, tie-pads, ballast and subgrade, but not the rail response. However, according to eq. (4), as well as Fig. 3,  $k$  does depend on the rail size; although this dependence is very small.

In situations when a locomotive or car on *three-axle trucks* is to be used as a test vehicle, eq. (6) has to be expanded, by including the effect of the additional axles. Denoting the wheel loads, shown in Fig. 4, as

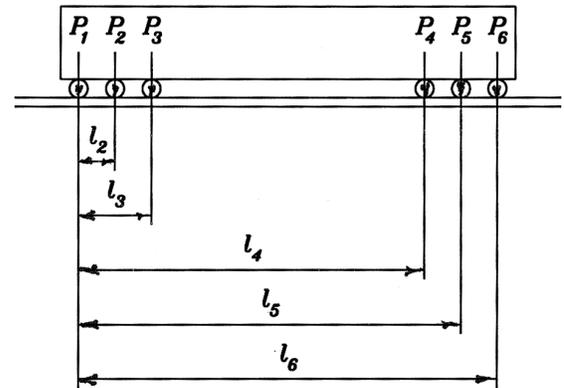


Fig. 4 Car or Loc on 3-Axle Trucks

$$\left. \begin{array}{lll} P_1 = P & ; & P_2 = n_2 P & ; & P_3 = n_3 P \\ P_4 = n_4 P & ; & P_5 = n_5 P & ; & P_6 = n_6 P \end{array} \right\} \quad (10)$$

where the  $n_2, \dots, n_6$  values are obtained by using a weighing scale and setting  $w(0) = w_m$ , the formula that corresponds to eq. (6) becomes

$$\begin{aligned} \frac{w_m}{P} = \frac{\beta}{2k} & \left[ 1 + n_2 e^{-\beta l_2} (\cos \beta l_2 + \sin \beta l_2) + n_3 e^{-\beta l_3} (\cos \beta l_3 + \sin \beta l_3) \right. \\ & + n_4 e^{-\beta l_4} (\cos \beta l_4 + \sin \beta l_4) + n_5 e^{-\beta l_5} (\cos \beta l_5 + \sin \beta l_5) \\ & \left. + n_6 e^{-\beta l_6} (\cos \beta l_6 + \sin \beta l_6) \right] \end{aligned} \quad (11)$$

noting again that  $l_1 = 0$ .

Next, the above equation has to be evaluated numerically for various rail sizes and a range of  $k$ -values, as done previously with eq. (6). The results of this evaluation are to be plotted as graphs in a master chart, similar to the one shown in Fig. 3. The procedure for determining the rail support modulus  $k$  is as before: First roll the test car on three-axle trucks to the location of interest, next measure the vertical deflection  $w_m$  at the first wheel with load  $P_1 = P$  shown in Fig. 4, then form  $w_m/P$  and get the  $k$ -value from the corresponding graph.

Note, that eq. (11) was derived for the case when the wheel deflection is measured at the first or the last wheel of the locomotive or car. Should it be planned instead to record deflections at any of the other wheels, then eq. (11) has to be modified accordingly.

From the above presentation it follows that *for the determination of the rail support modulus  $k$ , any car or locomotive may be utilized as a loading device and that only one measured rail deflection,  $w_m$ , is required.* The proposed method avoids the numerical solution of the involved transcendental equation (6) or (11) for the unknown  $k$ . It requires only the numerical evaluation of the right hand side of the corresponding equation for various  $k$ -values, which may be easily performed even on a programmable pocket calculator. The mobility of the chosen car or locomotive and the simplicity of determining the rail support modulus from one measured deflection  $w_m$  and a graph of the type shown in Fig. 3, allows for a rapid and economical determination of the track modulus  $k$  at various track locations.

#### **4. Other Proposed Methods for Determination of $k$**

At this stage, it is instructive to discuss other methods for the determination of  $k$  that were proposed in the railroad literature, but are difficult to use or are of questionable validity.

In *one of these methods*, the field test consisted of loading vertically only *one tie* that was separated from the rails by removing the fasteners, then by recording the vertical displacement of this tie, and by calculating the base parameter using the relation  $p(x) = kw(x)$  under the assumption that the tie-ballast pressure is uniform. In one test series, the loads were generated by a freight car of about 16 tons, which was equipped with two hydraulic jacks (one at each rail seat); a similar set-up to the one shown in Fig. 2. The jacks, when activated, pressed against the tie, lifting up the car; thus, exerting about 8 tons on each rail-seat. According to Driessen (1937), 385 tests of this type were conducted before World War II on the German, Dutch, and Swiss railroads for the purpose of determining the corresponding  $k$ -values. This effort was not successful, because it did not yield meaningful results. It is worth noting that tests of this type were conducted by the German Railways (DB) also after 1945, as described by Birmann (1957) and Nagel (1961).

It appears that the main problem with this method was that the used test, that loaded only one tie, has two major shortcomings. The *first* one is that because of the granular nature of the ballast and subgrade, their material properties may strongly vary along the track. Thus, the loading of *one* tie, at different locations along the track, will necessarily show a wide scatter in the obtained data. This is very apparent from the test data presented by Driessen (1937, page 123). The *second* shortcoming is that the base parameter  $k$ , that is a property of a layer of closely spaced individual springs, depends on the size of the loading area when used for a continuous base consisting of ballast and subgrade [For a recent proof of this assertion refer to Kerr (1987, page 40)]. Thus, the test that uses only one tie will not yield the same parameter  $k$  as when loading a row of closely spaced ties encountered in an actual track. In this connection note that according to Wasiutynski (1937), the  $k$ -value obtained when loading only one tie is about twice as large as when using the actual rail-tie structure. The above discussion suggests that, for the determination of  $k$ , the use of tests that load only one separated tie should be avoided.

Another method for the determination of  $k$  was proposed and used by the Talbot Committee (1918) and by Wasiutynski (1937). In this method a car is moved to the track location of interest, and the caused vertical rail deflections at each tie are measured, as shown in Fig. 5.

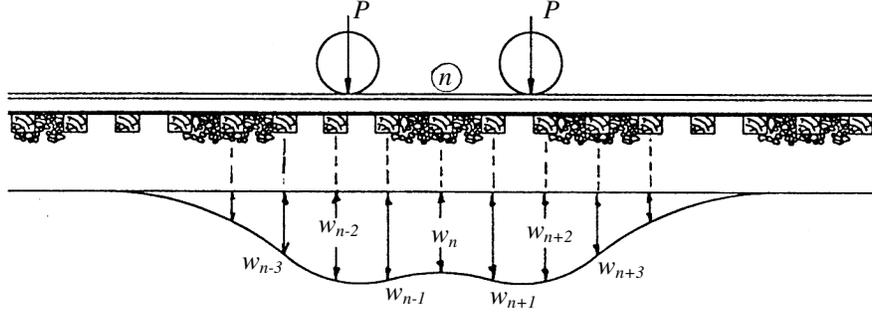


Fig. 5 Recorded Rail Deflections in Depressed Region

According to the Talbot Committee (1918) the rail support modulus  $k$  is then calculated by dividing the sum of the wheel loads  $\sum P$  that act on one rail, by the area formed between the undeformed straight rail and the deflected rail,  $A_R$ .

This prescription for the determination of  $k$  may be derived from vertical equilibrium of a rail. Noting that  $p(x)$  is the pressure that acts on the rail base (positive upwards), it follows that

$$\sum P - \int_{-\infty}^{\infty} p(x) dx = 0 . \quad (12)$$

Noting that  $p(x) = kw(x)$ , where  $k$  is constant along the track, and by definition is valid for one rail only, above equation becomes

$$\sum P - k \int_{-\infty}^{\infty} w(x) dx = 0 . \quad (13)$$

Solving for  $k$  we obtain

$$k = \frac{\sum P}{\int_{-\infty}^{\infty} w(x) dx} . \quad (14)$$

Since the integral in the denominator is the area formed by the deflected rail,  $A_R$ , the above  $k$ -expression proves that the prescription by the Talbot Committee satisfies vertical equilibrium.

However, early tests conducted by the Talbot Committee (1918, Section IV) revealed that the vertical rail deflections were not increasing linearly with increasing wheel loads, especially for tracks in poor condition. A similar type of non-linear response was recorded more recently by Zarembski and Choros (1980), for track in good condition but for larger wheel loads.

The observed non-linearity for relatively light wheel loads was attributed mainly to the play between the rails and the ties, the play between the ties and ballast, and the bending of the ties while they take full bearing in the ballast. For heavy wheel loads, an additional contributor to the non-linear response is the stiffening of the track caused by the increasing compression of the ballast and subgrade layers.

To take into consideration this non-linearity, in a later paper the Talbot Committee (1933, Section 37) recommended to retain the linear analysis based on eq. (1), but to determine the rail support modulus,  $k$ , using the difference between the vertical deflections from a heavy and a light car; thus, using the reduced shaded area shown in Fig. 6. For the determination of  $k$ , they

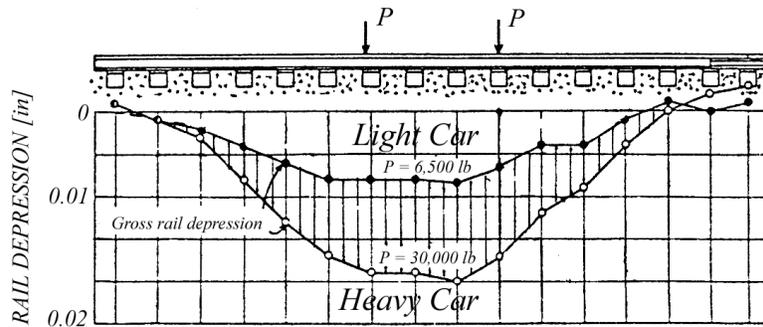


Fig. 6 Reduced Deflection Area

proposed the formula

$$k = \frac{\sum (P_h - P_l)}{a \sum_{i=1}^m (w_i^h - w_i^l)} \quad (15)$$

where  $a$  is the center-to-center tie spacing and the superscript  $h$  corresponds to heavy and  $l$  to

light wheels. The given justification of this formula was that the light wheel loads will eliminate the slack at all ties in the depressed track region and that further rail deflections, beyond those caused by the light wheel loads, will be proportional to the additional loads generated by the heavy wheels. For additional details, refer to Kerr and Shenton (1985). This method was used since then by many railway engineers and researchers. However, it is conceptually incorrect, as explained next.

To demonstrate this point consider, as example, the three *rail-tie contact pressure vs. rail deflection* curves at a point  $x=0$ , as shown in Fig. 7. At first, assume that each rail is pre-loaded by a uniformly

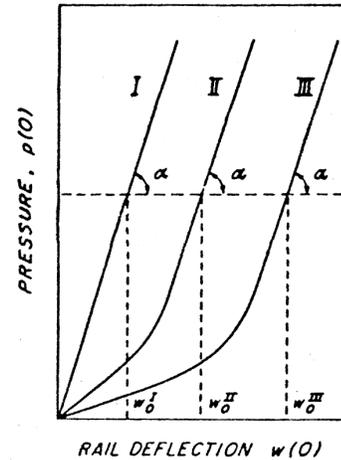


Fig. 7 Contact Pressure vs. Rail Deflections

distributed vertical load, as indicated by the horizontal dashed line. The resulting vertical displacements for each rail are uniform, but they differ in magnitude for each of the cases I, II, and III. In all three cases *no* bending moments are generated in the rails. Then, each rail is subjected *additionally* to a wheel load  $P$ . Each rail will respond linearly with  $k = \tan \alpha$ , and the rail deflections and bending moments caused by this additional load  $P$ , will be the same for all three cases.

However, when each rail is subjected *only* to a heavy wheel load  $P$  (without a large uniform pre-loading) the resulting deflections and bending moments will strongly differ from the ones described above. This was shown analytically by Kerr and Shenton (1986). Thus, when considering rails subjected to wheel loads whose base exhibits a non-linear response as shown in Fig. 7, a situation encountered especially on freight lines, the "soft" part of this response should *not* be neglected; otherwise, the determined rail support modulus will be too high.<sup>1)</sup>

Since the "reduced area" method described above requires many rail deflection measurements, recently Selig and Li (1994) proposed to simplify the determination of  $k$  by conducting a test that uses a single increasing wheel load and generates a load-deflection curve

<sup>1)</sup> This comment also applies to the tie-pad test, as specified in the AREA Manual (1993, Chapter 1.9.1.15c)

for one point, of type III in Fig. 7. They proposed to determine the rail support modulus as  $k = \tan \alpha$ , where  $\alpha$  is the angle of the steep part. This method for the determination of  $k$  is not correct either, since it neglects the "soft" part of the response curve, and thus will result in a  $k$ -value that is too high.

In still *another approach* to determine the rail support modulus  $k$ , various researchers in North America and Europe assumed that the rail supporting base (consisting of pads, ties, ballast, and subgrade) may be represented by layers of springs each with a different stiffness, arranged in series, as shown schematically in Fig. 8. The resulting rail support modulus for the entire base is

$$k = \frac{1}{1/k_p + 1/k_t + 1/k_b + 1/k_s}, \quad (16)$$

where  $k_p$  is the corresponding stiffness of the pad (if used),  $k_t$  is the stiffness of the tie (due to the compressibility of wood in the rail-seat region and tie bending),  $k_b$  is the vertical stiffness of the ballast layer, and  $k_s$  is the stiffness of the subgrade. For a discussion of this method refer to Novichkov (1955), Luber (1962), Shchepotin (1964), Shakhunyants (1965), Birmann (1965-66), and Ahlbeck et al. (1978, pp. 242-243).

This approach, although intuitively appealing, is not practical for the determination of  $k$ , because the correlation of the response of a sample of (disturbed) ballast or subgrade tested in a lab with the corresponding  $k_b$  or  $k_s$  value for an actual track is not very reliable. Also, the ballast and subgrade properties generally vary along the track and their response may be non-linear. Therefore, this approach is not suitable for the determination of the  $k$ -values for actual tracks.

Finally, it is instructive to discuss the *method* for the determination of the track modulus used extensively in the *German language railroad literature*. For examples of this approach refer to the books by Hanker (1952, Section V.3.d), Schoen (1967, p. 263), Eisenmann in

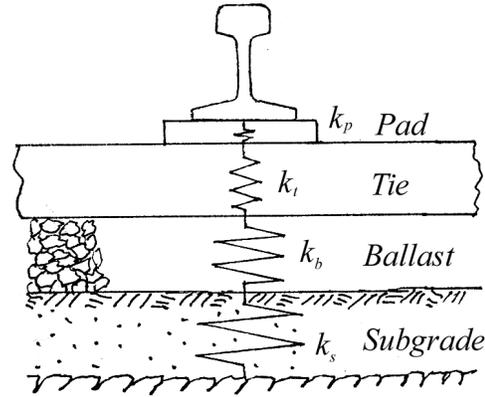


Fig. 8 Track Model Consisting of Spring Layers.

Fastenrath (1981, Part 2, Section 3.1) and Führer (1978, Section 3.1.2.1).

Their approach is based on the original assumption by Winkler (1867, § 195) for the *longitudinal-tie track*, that the contact pressure between tie and support is

$$p^*(x) = C w(x) , \quad (17)$$

where  $p^*$  has the dimension of force per unit area, and  $C$  is the base parameter that is independent of the tie width. Since, in the differential equation for a continuously supported beam

$$EI \frac{d^4 w}{dx^4} + p(x) = q(x) , \quad (18)$$

each term, including  $p(x)$ , is of the dimension force per unit length, Winkler defined

$$p(x) = b_o p^*(x) = b_o C w(x) , \quad (19)$$

where  $b_o$  is the width of the longitudinal-tie. Substituting it into eq. (18), Winkler obtained the differential equation

$$EI \frac{d^4 w}{dx^4} + b_o C w(x) = q(x) , \quad (20)$$

instead of eq. (1). Subsequently, this equation was adapted by Schwedler (1882) and it plays a key role in the often-quoted book by Zimmermann (1887, 1930, 1941), for longitudinal-tie tracks.

The multiplication by  $b_o$  in eq. (19), although valid for a Winkler base that consists of closely spaced *independent* springs, is of questionable validity when a longitudinal-tie rests on a *continuum* base made up of ballast and subgrade. This was shown by Kerr (1987, pp. 39-40).

When the German and Austrian railroad engineers adapted eq. (20) for the analysis of a *cross-tie track*, they

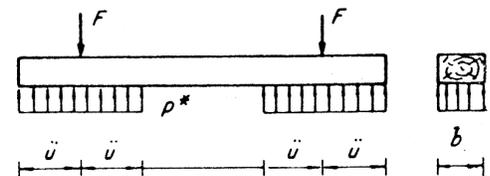


Fig. 9 The Saller Assumption

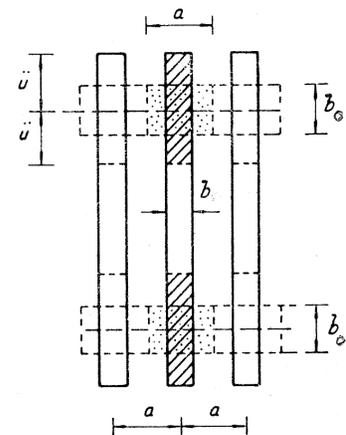


Fig.10 Hanker Transformation of Cross-Tie to Longitudinal Tie Track

faced the problem of choosing the two parameters  $C$  and  $b_o$ . Since the parameter  $C$  was assumed to be independent of the tie shape, they found it necessary to establish an "effective track width"  $b_o$  for the cross-tie track. Saller (1932), assumed that

$$b_o = \frac{2\ddot{u}b}{a}, \quad (21)$$

where  $\ddot{u}$  is the distance from the rail center to the end of tie,  $b$  is the width of a cross-tie, as shown in Fig. 9, and  $a$  is the center-to-center tie spacing, as shown in Fig. 10.

In an attempt to prove (or justify) the validity of Saller's assumption for the determination of  $b_o$ , stated in eq. (21), Hanker (1935) transformed the cross-tie track into a pseudo longitudinal-tie track in accordance with the scheme shown in Fig. 10. As part of this transformation, Hanker introduced a *condition* that the effective tie-ballast *contact areas* for both cases are to be *equal*. Namely, that

$$ab_o = 2\ddot{u}b \quad (22)$$

This condition solved for  $b_o$  yields directly the assumption (21) by Saller.

The above transformation, and eq. (21) for  $b_o$ , was generally accepted in the German language railroad literature. For examples refer to Hanker (1952), Schoen (1967), Eisenmann in Fastenrath (1981, Section 3.1), Führer (1978, Section 3.1.2) and Kaess and Gottwald (1979).

The condition of *equal contact areas* was apparently conceived by considerations of vertical equilibrium and by the notion that for a given rail-seat force the pressures in the effective tie-ballast contact areas should be constant and equal. Namely, that  $p_o ab_o = p_o 2\ddot{u}b$ . This is indeed the case when the rail support is represented by the Winkler base consisting of closely spaced, *independent*, springs. But, it is not true for an actual track base, as described previously. Therefore, the geometrical transformation shown in Fig. 10 does not correspond to an actual track situation and is of questionable validity for railroad engineering purposes.

The need to determine the "effective track width"  $b_o$ , arose from the use of differential equation (20) with the *a priori* assumption that there exists a constant parameter  $C$  for the rail supporting base. As shown by Kerr (1987), this is not the case for actual tracks. The

determination of the second parameter  $b_0$  is also of questionable validity, as discussed above. Therefore, the use of eq. (20) in conjunction with the two parameters  $b_0$  and  $C$  is not justified, and hence, not advisable.

Because, of the shortcomings of the reviewed methods for the determination of the rail support modulus, it is suggested that *for cross-tie tracks, differential equation (1) with the one base parameter  $k$  be used. This parameter should be determined from one field measurement using a test car on one or two axle trucks, as shown at the beginning of this paper.*

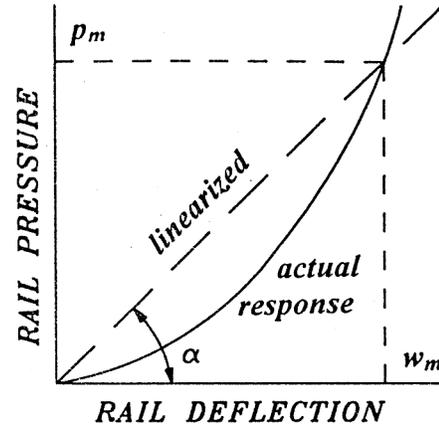


Fig. 11 Deflection vs. Pressure

### 5. Problems to be Considered When Determining $k$

As discussed previously, test revealed that the vertical track response is generally *non-linear*. However, the standard track analysis is based on the *linear* differential equation (1), where

$$k = \tan \alpha = \frac{p_m}{w_m} \quad (23)$$

as shown in Fig. 11. This  $k$ -value is determined using eq. (4) or graphs of the type presented in Fig. 3.

At this stage, it is necessary to clarify the effect of the magnitude of the test wheel loads on the determined  $k$ -value. When a light passenger car is used as a loading device, according to eq.(23) the corresponding  $k = \tan \alpha_1$ , whereas when a heavier freight car is used, the corresponding  $k = \tan \alpha_2$  as shown in Fig. 12; thus, a larger  $k$  - value results for the same track. Therefore, when the vertical track response is non-linear, *the weight*

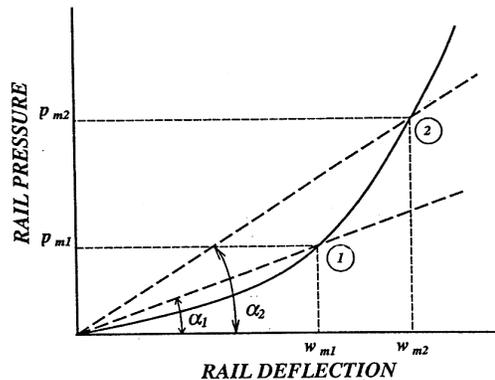


Fig.12 Rail Deflection vs. Rail Pressure

*of the loading vehicle should be as close as possible to those of the anticipated traffic.*

According to Mair (1976) and Kerr and Shenton (1986) this procedure yields *rail bending moments* that are on the safe side. However, as shown by Kerr and Shenton (1985, 1986), the corresponding calculated *rail seat force*,  $F_{\max}$ , used for the determination of the needed tie-plate size and the required depth of the ballast layer, is grossly underestimated. The "linear"  $F_{\max}$ -value has to be multiplied by a correction factor of **1.5**, in order to represent the actual field conditions.

There are three other *problems to be taken into consideration when determining  $k$* . They are: (1) The effect of continuously increasing rail deflections near the wheels after the test vehicle is stopped at the track location of interest, (2) The effect of thermal tension or compression forces in a CWR on the determined  $k$ -value using eq. (4) or the graphs of Fig. 3 that do *not* include axial forces, and (3) The effect of ballast disturbance on the rail support modulus  $k$ .

During some loading tests for the determination of  $k$  it was observed that after the test vehicle is placed on the track at the location of interest, in addition to the instantaneous rail deflections *the rails continued to deflect* as time progressed; especially in the vicinity of the wheel loads. In such cases the question arises as to when should  $w_m$  be recorded?

When the rails continue to deflect after the load is placed, this means that the base is not elastic. This is generally caused by a slow squeeze-out of the water that is trapped in a subgrade layer of poor permeability (like clay or silt). For these cases the elastic springs in the Winkler model, shown in Fig. 1, have to be augmented by including viscous elements, as shown in Fig. 13, and the resulting governing equations should then be solved for the prescribed wheel loads of interest.

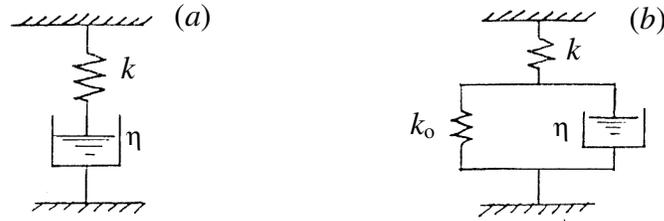


Fig 13 Foundation Models for Time Dependent Base Deflections.

Both models exhibit an instantaneous elastic deflection. However, in Case (a) the deflections continue for a long time (which may occur for very thick clay layers) whereas in Case (b) after a relatively short time the rate of the non-elastic deflections decreases substantially (which may occur for very thin clay layers).

Often, the track modulus  $k$  is needed for main line tracks that are subjected to *moving trains*. In these cases there is no sufficient time for the trapped water to be squeezed out, and the track will respond elastically for both cases shown in Fig. 13. The corresponding rail support modulus  $k$  is determined as discussed previously, by recording  $w_m$  *immediately after loading* and then utilizing Fig. 3.

When a train stops on the track for a prolonged period of time, with a base that responds like the models in Fig. 13, then the maximum rail deflections and bending moments, hence the rail stresses, will differ from the elastic case. The analysis of these cases is more involved and requires solutions for a rail on a corresponding visco-elastic base, as mentioned previously.

The second problem to be clarified is the *effect of axial forces in CWR's* on the determination of  $k$  caused, for example, by changes in the rail temperature. To do this, consider a rail-in-track subjected to a uniform tension force  $N_0$  and a wheel load  $P$ , as shown in Fig. 14.

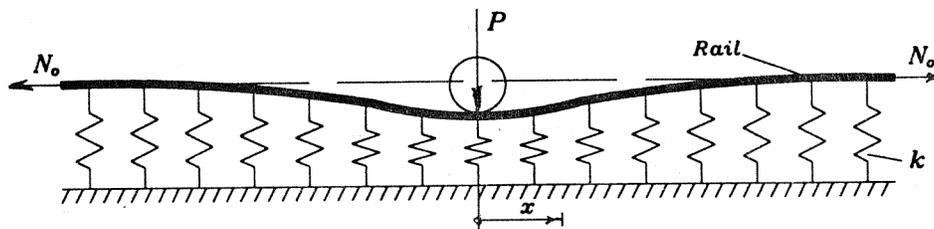


Fig. 14 Rail-in-Track Subjected to Axial Force  $N_0$  and Wheel Load  $P$ .

The governing differential equation for this rail is

$$EIw^{IV} - N_o w'' + kw = q \quad -\infty < x < \infty \quad (24)$$

where  $N_o w''$  is the term added to eq. (1) in order to include the effect of the axial tension force.

The resulting deflection at the wheel load is [Hetényi (1947), Chapter VI, p.129]

$$w(0) = \frac{P}{2k} \frac{\sqrt{k/(4EI)}}{\sqrt{\sqrt{k/(4EI)} + N_o/(4EI)}} \quad (25)$$

Setting  $w(0) = w_m$  it follows that

$$\frac{w_m}{P} = \frac{1}{2k} \frac{\sqrt{k/(4EI)}}{\sqrt{\sqrt{k/(4EI)} + N_o/(4EI)}} \quad (26)$$

When  $N_o = 0$ , the above equation reduces to eq. (3), as expected. When  $N_o$  is a compression force,  $N_o$  is replaced by  $(-N_o)$  in the above equations [Hetényi (1947), p.135].

Eq. (26) was evaluated for a 115 RE rail,  $N_o = 0$  and  $N_o = \pm 50$  tons, and a range of  $k$ -values. The results are shown in Fig.15

Noting that  $N_o = 50$  tons (100,000 lb) corresponds to a temperature change of about

40° F from neutral<sup>1)</sup>, it is concluded that for *the anticipated range of temperature changes, the axial force  $N_o$  has a negligible effect on the determined  $k$ -value, using eq. (4).* This also applies to Fig. 3 for test cars with two-axle trucks. Thus, tests for the determination of the track modulus  $k$  for CWR tracks may be conducted at any reasonable ambient temperature.

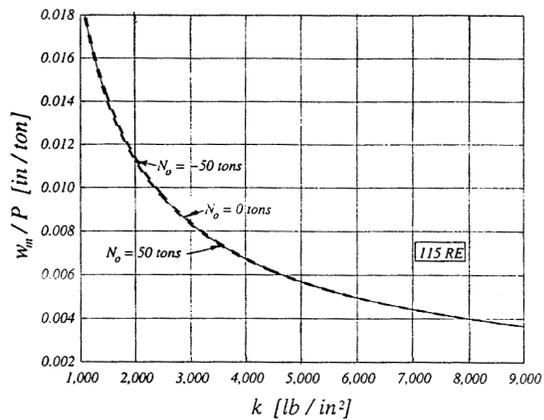


Fig. 15 Effect of  $N_o$  on  $w_m/P$  vs. Rail Support Modulus  $k$

<sup>1)</sup> The *neutral temperature* for CWR's is the rail temperature at which the axial forces in the rails are zero. In North America it ranges generally from 85°F to 115°F. It varies with the geographical location of the track territory under consideration; the high temperatures are for the southern USA, in order to avoid track buckling (

When determining the rail support modulus  $k$ , it should be noted that if the *ballast-in-track is disturbed* in the region of interest (for example, by tamping after timbering and surfacing or by spot renewal), the  $k$ -value diminishes, as shown schematically in Fig. 16.

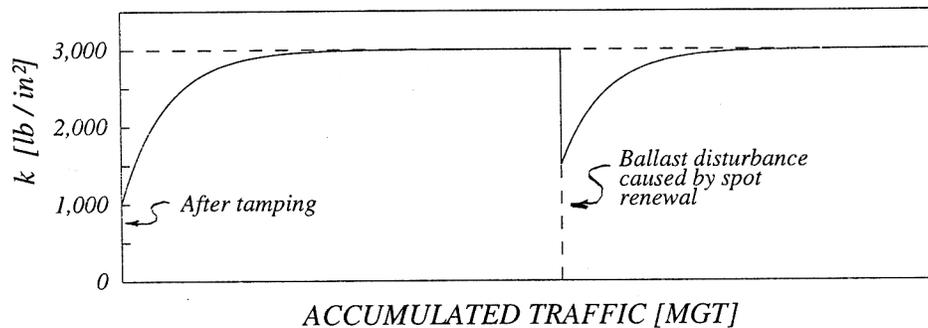


Fig. 16 Effect of Ballast Disturbance on  $k$  of a Wood Tie Track at a Track Location  
 Typical  $k$ -values at a track location, as affected by a ballast disturbance and then by the accumulated tonnage of moving trains, are shown in Fig. 16 for a wood-tie track. Note, whereas a track disturbance lowers the  $k$ -value, the moving traffic tends to restore it.

These track responses should be taken into consideration when attempting to determine the  $k$ -value at a specific location of a railroad track.

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